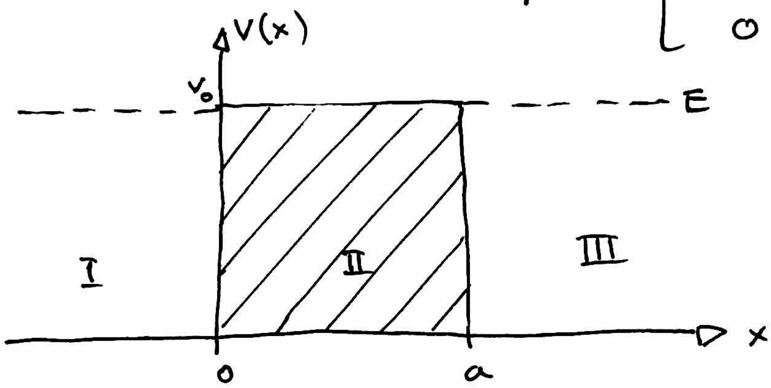
Exempeltenta FUF040

Uppgift 1 En partikel med massa m och energi $E = V_0$ i potentialen

$$V(x) = \begin{cases} 0, & x < 0 \\ V_0, & 0 \le x \le a \\ 0, & x > a \end{cases}$$



$E = V_0 > 0$

a) Schrödingerekvationen:

$$\hat{H}\psi(x) = \left(\frac{\hat{p}^2}{2m} + V(x)\right)\psi(x) =$$

$$= \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\psi(x)$$

• $\underline{x < 0}$: $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$

$$\Rightarrow \begin{cases} \psi_I(x) = A_+ e^{ikx} + A_- e^{-ikx} \\ k = \frac{1}{\hbar}\sqrt{2mE} \end{cases}$$

• $\underline{0 \le x \le a}$: $\frac{d^2\psi}{dx^2} = 0 \Rightarrow \psi_{II}(x) = \alpha x + \beta$

• $\underline{x > a}$: som för $x < 0 \Rightarrow$

$$\begin{cases} \psi_{III}(x) = B_+ e^{ikx} + B_- e^{-ikx} \\ k = \frac{1}{\hbar}\sqrt{2mE} \end{cases}$$

b) Randvillkor:

$$\begin{cases} \Psi_I(0) = \Psi_{II}(0) \Rightarrow & \beta = A_+ + A_- \\ \Psi_{II}(a) = \Psi_{III}(a) \Rightarrow & \alpha a + \beta = B_+ e^{ika} + B_- e^{-ika} \\ \dfrac{d\Psi_I(0)}{dx} = \dfrac{d\Psi_{II}(0)}{dx} \Rightarrow & \alpha = ik(A_+ - A_-) \\ \dfrac{d\Psi_{II}(a)}{dx} = \dfrac{d\Psi_{III}(a)}{dx} \Rightarrow & \alpha = ik(B_+ e^{ika} - B_- e^{-ika}) \end{cases}$$

Lös för $A_+ + A_-$ och $A_+ - A_-$:

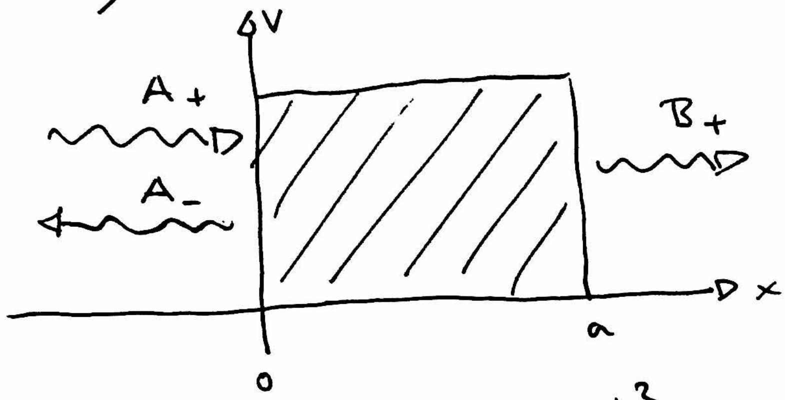
$$\begin{cases} A_+ + A_- = (1-ika)B_+ e^{ika} + (1+ika)B_- e^{-ika} \\ A_+ - A_- = B_+ e^{ika} - B_- e^{-ika} \end{cases}$$

$$2A_+ = (2-ika)B_+ e^{ika} + ikaB_- e^{-ika}$$

$$2A_- = -ikaB_+ e^{ika} + (2+ika)B_- e^{-ika}$$

$$\therefore \begin{cases} A_+ = \left(1 - \dfrac{ika}{2}\right)B_+ e^{ika} + \dfrac{ika}{2}B_- e^{-ika} \\ A_- = -\dfrac{ika}{2}B_+ e^{ika} + \left(1 + \dfrac{ika}{2}\right)B_- e^{-ika} \end{cases}$$

c) Transmissionskoefficienten för $B_- = 0$



$$T \equiv \frac{|B_+|^2}{|A_+|^2} = \frac{|B_+|^2}{\left|\left(1-\frac{ika}{2}\right)B_+\right|^2}$$

$$= \frac{1}{1 + k^2a^2/4}$$

<u>check</u>: – om a växer så minkar T.
verkar rimligt
– om V_0 (dvs E) minskar $\Rightarrow$
k minskar $\Rightarrow$ T ökar.
verkar rimligt

Uppgift 2

Kreations- och annihilationsoperatorerna ges av

$$\begin{cases} \hat{a}_+ = \dfrac{1}{\sqrt{2\hbar m\omega}}\left(-i\hat{p} + m\omega\hat{x}\right) \\ \hat{a}_- = \dfrac{1}{\sqrt{2\hbar m\omega}}\left(i\hat{p} + m\omega\hat{x}\right) \end{cases}$$

Då har vi att

$$\begin{cases} \hat{x} = \sqrt{\dfrac{\hbar}{2m\omega}}\left(\hat{a}_+ + \hat{a}_-\right) \\ \hat{p} = i\sqrt{\dfrac{\hbar m\omega}{2}}\left(\hat{a}_+ - \hat{a}_-\right) \end{cases}$$

Väntevärden:

$$\langle \hat{x} \rangle \equiv \langle \Psi_n | \hat{x} | \Psi_n \rangle =$$

$$= \langle \Psi_n | \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}_+ + \hat{a}_-) | \Psi_n \rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \Big(\langle \Psi_n | \hat{a}_+ | \Psi_n \rangle + \langle \Psi_n | \hat{a}_- | \Psi_n \rangle \Big)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \Big[\langle \Psi_n | \sqrt{n+1} | \Psi_{n+1} \rangle + \langle \Psi_n | \sqrt{n} | \Psi_{n-1} \rangle \Big]$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \Big[\sqrt{n+1} \langle \Psi_n | \Psi_{n+1} \rangle + \sqrt{n} \langle \Psi_n | \Psi_{n-1} \rangle \Big]$$

Men $\langle \Psi_n | \Psi_m \rangle = \delta_{nm} \Rightarrow$

$$\boxed{\langle x \rangle = 0}$$

$$\langle \hat{p} \rangle = \langle \Psi_n | \hat{p} | \Psi_n \rangle = \langle \Psi_n | i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-) | \Psi_n \rangle$$

$$= i\sqrt{\frac{\hbar m\omega}{2}}\Big[\langle \Psi_n | \hat{a}_+ | \Psi_n \rangle - \langle \Psi_n | \hat{a}_- | \Psi_n \rangle\Big]$$

$$= i\sqrt{\frac{\hbar m\omega}{2}}\Big[\langle \Psi_n | \sqrt{n+1} | \Psi_{n+1} \rangle - \langle \Psi_n | \sqrt{n} | \Psi_{n-1} \rangle\Big]$$

$$= i\sqrt{\frac{\hbar m\omega}{2}}\Big[\sqrt{n+1}\,\langle \Psi_n | \Psi_{n+1} \rangle - \sqrt{n}\,\langle \Psi_n | \Psi_{n-1} \rangle\Big]$$

$\langle \Psi_n | \Psi_m \rangle = \delta_{nm}$

$$= 0$$

$$\langle x^2 \rangle = \langle \Psi_n | \hat{x}^2 | \Psi_n \rangle =$$

$$= \langle \Psi_n | \frac{\hbar}{2m\omega}(\hat{a}_+ + \hat{a}_-)(\hat{a}_+ + \hat{a}_-) | \Psi_n \rangle$$

$$= \frac{\hbar}{2m\omega}\Big[\langle \Psi_n | \hat{a}_+ \hat{a}_- | \Psi_n \rangle + \langle \Psi_n | \hat{a}_- \hat{a}_+ | \Psi_n \rangle\Big]$$

där vi ~~använt~~ ~~att~~ $\langle \Psi_n | \Psi_m \rangle = \delta_{nm}$

$$\Rightarrow \langle x^2 \rangle = \frac{\hbar}{2m\omega}\Big[\langle \Psi_n | \hat{a}_+ \sqrt{n} | \Psi_{n-1} \rangle + \langle \Psi_n | \hat{a}_- \sqrt{n+1} | \Psi_{n+1} \rangle\Big]$$

$$= \frac{\hbar}{2m\omega}\Big[\sqrt{n}\,\langle \Psi_n | \sqrt{n} | \Psi_n \rangle + \sqrt{n+1}\,\langle \Psi_n | \sqrt{n+1} | \Psi_n \rangle\Big]$$

$$\Rightarrow \quad \underline{\langle x^2\rangle} = \frac{\hbar}{2m\omega}\left[n \underbrace{\langle \psi_n | \psi_n\rangle}_{\equiv 1} + (n+1)\langle \psi_n|\psi_n\rangle \right]$$

$$= \frac{\hbar}{2m\omega}(2n+1) = \underline{\underline{\frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)}}$$

$$\langle \hat{p}^2\rangle = \langle \psi_n | \hat{p}^2 | \psi_n\rangle =$$

$$= \langle \psi_n | -\frac{\hbar m\omega}{2}(\hat{a}_+ - \hat{a}_-)(\hat{a}_+ - \hat{a}_-)|\psi_n\rangle$$

$$= -\frac{\hbar m\omega}{2}\langle \psi_n | -\hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+ | \psi_n\rangle$$

där vi använt $\langle \psi_n | \psi_m\rangle = \delta_{mn} \Rightarrow$

$$\underline{\langle \hat{p}^2\rangle} = \frac{\hbar m\omega}{2}\langle \psi_n | \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ | \psi_n\rangle$$

$$= \frac{\hbar m\omega}{2}\left(n\langle \psi_n|\psi_n\rangle + (n+1)\langle\psi_n|\psi_n\rangle\right)$$

$$= \frac{\hbar m\omega}{2}(2n+1)$$

$$= \underline{\underline{\hbar m\omega\left(n+\frac{1}{2}\right)}}$$

<u>Osäkerhetsrelationen:</u>

$$\sigma_x^2 = \langle (\hat{x} - \langle x\rangle)\psi_n | (\hat{x} - \langle x\rangle)\psi_n\rangle$$

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)}$$

$$\sigma_p^2 = \langle (\hat{p} - \langle p\rangle)\psi_n | (\hat{p} - \langle p\rangle)\psi_n\rangle$$

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\hbar m\omega\left(n+\frac{1}{2}\right)}$$

$$\Rightarrow \quad \sigma_x\sigma_p = \sqrt{\frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)\hbar m\omega\left(n+\frac{1}{2}\right)}$$

$$= \hbar\left(n+\frac{1}{2}\right) \geq \frac{\hbar}{2}$$

Uppgift 3

En väteatom befinner sig i det mixade tillståndet

$$\Psi(r,\theta,\phi,t) = N\left(3e^{-iE_1 t/\hbar}\psi_{100} + e^{-iE_2 t/\hbar}\psi_{200}\right)$$

med $\psi_{n\ell m} = R_{n\ell}(r)Y_\ell^m(\theta,\phi)$ och E_1, E_2 är de två respektive lägsta energinivåerna.

a) Normalisering:

$$1 = \int |\Psi|^2 d^3\vec{r} = |N|^2\Big[\int 9\psi_{100}^*\psi_{100}\,d^3\vec{r} + 3e^{i(E_1-E_2)t/\hbar}\int \psi_{100}^*\psi_{200}\,d^3\vec{r} + 3e^{-i(E_1-E_2)t/\hbar}\int \psi_{200}^*\psi_{100}\,d^3\vec{r} + \int \psi_{200}^*\psi_{200}\,d^3\vec{r}\Big]$$

ortonormalitet

$$\overset{\downarrow}{=} |N|^2(9+0+0+1) = 10|N|^2$$

$$\Rightarrow \boxed{N = \frac{1}{\sqrt{10}}}$$

b) Väntevärdet av energin:

$$\hat{H}\Psi = \frac{3}{\sqrt{10}} e^{-iE_1t/\hbar} \hat{H}\psi_{100} + \frac{1}{\sqrt{10}} e^{-iE_2t/\hbar} \hat{H}\psi_{200}$$

$$= \frac{3}{\sqrt{10}} e^{-iE_1t/\hbar} E_1\psi_{100} + \frac{1}{\sqrt{10}} e^{-iE_2t/\hbar} E_2\psi_{200}$$

$$\langle E \rangle = \langle \Psi | \hat{H}\Psi \rangle =$$

$$= \frac{9}{10} E_1 \langle \psi_{100} | \psi_{100} \rangle + 0 + 0$$

$$+ \frac{1}{10} E_2 \langle \psi_{200} | \psi_{200} \rangle$$

$$= \frac{9}{10} E_1 + \frac{1}{10} E_2$$

Men $E_n \propto \frac{1}{n^2} \Rightarrow E_2 = \frac{E_1}{4}$

$$\Rightarrow \boxed{\langle E \rangle = \frac{9}{10} E_1 + \frac{1}{10} \cdot \frac{E_1}{4} = \frac{37}{40} E_1}$$

c) Väntevärdet på radien av elektronbanan:

$$\langle r \rangle = \langle \Psi | r \Psi \rangle = \int \Psi^* r \Psi \, d^3\vec{r}$$

$$= \frac{1}{10} \Big[9 \int \psi_{100}^* r \psi_{100} \, d^3\vec{r} + 3 e^{-i(E_2 - E_1)t/\hbar} \int \psi_{100}^* r \psi_{200} \, d^3\vec{r}$$

$$+ 3 e^{-i(E_1 - E_2)t/\hbar} \int \psi_{200}^* r \psi_{100} \, d^3\vec{r}$$

$$+ \int \psi_{200}^* r \psi_{200} \, d^3\vec{r} \Big]$$

• Första termen:

$$\int \Psi_{100}^* r \Psi_{100}\, d^3\vec{r} = \int R_{10}^* R_{10} r^3 dr \int Y_0^{0*} Y_0^0 \, d\Omega$$

$$= \int R_{10}^* R_{10}\, r^3 dr = \frac{4}{a^3}\int r^3 e^{-2r/a}\, dr$$

$$= \frac{4}{a^3}\left(\frac{a}{2}\right)^4 3! = \frac{3}{2}a$$

• Andra termen:

$$\int \Psi_{100}^* r \Psi_{200}\, d^3\vec{r} = \int R_{10}^* R_{20}\, r^3 dr \int Y_0^{0*} Y_0^0 \, d\Omega$$

$$= \int R_{10}^* R_{20} r^3 dr = \frac{\sqrt{2}}{a^3}\left[\int e^{-\frac{3r}{2a}} r^3 dr\right.$$

$$\left. - \frac{1}{2a}\int e^{-\frac{3r}{2a}} r^4 dr\right]$$

$$= -\sqrt{2}\, a\, \frac{32}{81}$$

• Tredje termen:

$$\int \Psi_{200}^* r \Psi_{100}\, d^3\vec{r} = \int R_{20}^* R_{10}\, r^3 dr \int Y_0^{0*} Y_0^0 \, d\Omega$$

$$= \int R_{20}^* R_{10}\, r^3 dr = -\sqrt{2}\, a\, \frac{32}{81}$$

• Fjärde termen:

$$\int \Psi_{200}^* r \Psi_{200}\, dr = \int R_{20}^* r^3 R_{20}\, dr \int Y_0^{0*} Y_0^0 \, d\Omega$$

$$= \int R_{20}^* R_{20}\, r^3 dr = \frac{1}{2a^3}\left[\int r^3 e^{-r/a} dr - \right.$$

$$\left. - \frac{1}{a}\int r^4 e^{-r/a} dr + \frac{1}{4a^2}\int r^5 e^{-r/a} dr\right]$$

$$\Rightarrow \int \psi_{200}^{*}\, r\, \psi_{200}\, d^3\vec{r} = \frac{a}{2}\cdot 12 = 6a$$

$$\Rightarrow \langle r \rangle = \frac{39}{20}a - \sqrt{2}\,a\,\frac{64}{270}\cos\left(\frac{(E_1 - E_2)t}{\hbar}\right)$$

Uppgift 4

a) Hermitisk operator: $\hat{Q}^\dagger = \hat{Q}$

Egenvärde: $\hat{Q}\Psi = q\Psi$

$$\langle Q \rangle = \langle \Psi | \hat{Q}\Psi \rangle = \langle \Psi | q\Psi \rangle$$
$$= q\langle \Psi | \Psi \rangle = q$$

Men $\langle Q \rangle = \langle \hat{Q}^\dagger \Psi | \Psi \rangle \equiv \langle \hat{Q}\Psi | \Psi \rangle$
$$= q^* \langle \Psi | \Psi \rangle = q^*$$
$$\Rightarrow \boxed{q = q^*}$$

b) $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$ hermitisk?

$$\langle \Psi | \hat{H}\Psi \rangle = \langle \Psi | \left(\frac{\hat{p}^2}{2m} + V(x)\right)\Psi \rangle$$
$$= \int \Psi^* \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\Psi \, dx$$
$$= \int \Psi^* \left(-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + V(x)\Psi\right) dx$$
$$= -\frac{\hbar^2}{2m}\left[\int \frac{d}{dx}\left(\Psi^* \frac{d\Psi}{dx}\right) dx - \int \frac{d\Psi^*}{dx}\frac{d\Psi}{dx} dx\right]$$
$$+ \int (V\Psi)^* \Psi \, dx$$
$$= -\frac{\hbar^2}{2m}\left[\cancel{\Psi^* \frac{d\Psi}{dx}}\Big|_{-\infty}^{+\infty} - \int \frac{d\Psi^*}{dx}\frac{d\Psi}{dx} dx\right]$$
$$+ \int (V\Psi)^* \Psi \, dx$$
$$= -\frac{\hbar^2}{2m}\left[- \int \frac{d}{dx}\cancel{\left(\frac{d\Psi^*}{dx}\Psi\right)} dx + \int \frac{d^2\Psi^*}{dx^2}\Psi \, dx\right]$$
$$+ \int (V\Psi)^* \Psi \, dx$$
$$= \int \left[\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V\right)\Psi\right]^* \Psi \, dx = \underline{\underline{\langle \hat{H}\Psi | \Psi \rangle}}$$

c) $[x^n, \hat{p}]f(x) = (x^n\hat{p} - \hat{p}x^n)f(x)$

$= \left[x^n\left(-i\hbar\frac{d}{dx}\right) - \left(-i\hbar\frac{d}{dx}\right)x^n\right]f(x)$

$= -i\hbar\left[x^n\frac{d}{dx} - \frac{d}{dx}x^n\right]f(x)$

$= -i\hbar\left[x^n\frac{d}{dx} - nx^{n-1} - x^n\frac{d}{dx}\right]f(x)$

$= i\hbar n x^{n-1} f(x)$

$\therefore \boxed{[x^n, \hat{p}] = i\hbar n x^{n-1}}$

Uppgift 5

a) Normalisering: $\chi = A\begin{pmatrix}3i\\4\end{pmatrix}$

$$1 = \chi^{\dagger}\chi = A^*(-3i, 4)A\begin{pmatrix}3i\\4\end{pmatrix}$$

$$= |A|^2(9+16) = 25|A|^2$$

$$\Rightarrow \boxed{|A| = \frac{1}{5}}$$

b) Egentillstånd till

$$S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

Två egentillstånd ($\chi_+^{(y)}$ och $\chi_-^{(y)}$).

$$\hat{S}_y \chi_+^{(y)} = \frac{\hbar}{2} c \chi_+^{(y)}$$

$$\rightarrow \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\alpha_+\\ \beta_+\end{pmatrix} = \frac{\hbar}{2} c\begin{pmatrix}\alpha_+\\ \beta_+\end{pmatrix}$$

där $\chi_+^{(y)} \equiv \begin{pmatrix}\alpha_+\\ \beta_+\end{pmatrix} \Rightarrow$

$$\begin{pmatrix}-i\beta_+\\ i\alpha_+\end{pmatrix} = \begin{pmatrix}c\alpha_+\\ c\beta_+\end{pmatrix} \Rightarrow \begin{cases}c\alpha_+ = 1\\ c\beta_+ = i\end{cases}$$

$$\therefore \chi_+^{(y)} = c\begin{pmatrix}1\\ i\end{pmatrix}$$

Normalisera: $(\chi_+^{(y)})^{\dagger}\chi_+^{(y)} =$

$$= c^*(1, -i)\, c\begin{pmatrix}1\\ i\end{pmatrix}$$

$$= |c|^2(1+1) = 2|c|^2 \equiv 1$$

$$|c| = \frac{1}{\sqrt{2}} \Rightarrow$$

$$\chi_+^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$$

På samma sätt för $\chi_-^{(y)} \Rightarrow$

$$\chi_-^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}.$$

c) $\chi = a\chi_+^{(y)} + b\chi_-^{(y)} \equiv \frac{1}{5}\begin{pmatrix}3i\\4\end{pmatrix}$

där $a = \langle \chi_+^{(y)} | \chi \rangle =$

$$= \frac{1}{\sqrt{2}}(1, -i)\left(\frac{1}{5}\right)\begin{pmatrix}3i\\4\end{pmatrix} = \frac{-i}{5\cdot\sqrt{2}}$$

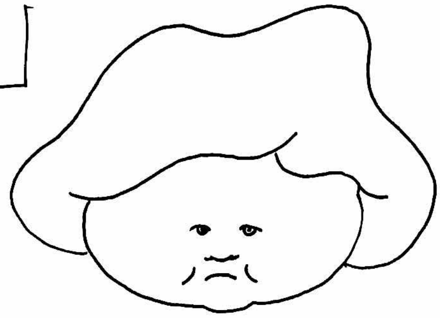
$$b = \langle \chi_-^{(y)} | \chi \rangle = \frac{7i}{5\sqrt{2}}$$

$$\therefore \boxed{\chi = \frac{-i}{5\sqrt{2}}\chi_+^{(y)} + \frac{7i}{5\sqrt{2}}\chi_-^{(y)}}$$

d)
$$\begin{cases} |b|^2 = \left|\frac{7i}{5\sqrt{2}}\right|^2 = \frac{49}{50} \\ |a|^2 = \left|\frac{-i}{5\sqrt{2}}\right|^2 = \frac{1}{50} \end{cases}$$

$$\therefore \boxed{|b|^2 = \frac{49}{50}}$$

Uppgift 6

a) 

b) För en observabel Q gäller

$$\frac{d}{dt}\langle Q\rangle \equiv \frac{d}{dt}\langle \Psi | \hat{Q}\Psi\rangle =$$

$$= \left\langle \frac{\partial \Psi}{\partial t} \middle| \hat{Q}\Psi \right\rangle + \left\langle \Psi \middle| \frac{\partial \hat{Q}}{\partial t}\Psi \right\rangle$$

$$+ \left\langle \Psi \middle| \hat{Q}\frac{\partial \Psi}{\partial t}\right\rangle .$$

Med Schrödingerekvationen $i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$ får vi då:

$$\frac{d}{dt}\langle Q\rangle = -\frac{1}{i\hbar}\langle \hat{H}\Psi | \hat{Q}\Psi\rangle$$

$$+ \frac{1}{i\hbar}\langle \Psi | \hat{Q}\hat{H}\Psi\rangle$$

$$+ \left\langle \frac{\partial Q}{\partial t}\right\rangle$$

$$= \frac{i}{\hbar}\langle [\hat{H},\hat{Q}]\rangle + \left\langle \frac{\partial Q}{\partial t}\right\rangle$$

eftersom $\hat{H}$ är hermitsk.

Låt nu $Q \to L_x, L_y, L_z$

(ej tidsberoende! $\vec{L} = \vec{r}\times\vec{p}$) $\Rightarrow$

$$\frac{d\langle L_x\rangle}{dt} = \frac{i}{\hbar}\langle [\hat{H},\hat{L}_x]\rangle$$

etc.

För $\hat{H} = \frac{\hat{p}^2}{2m} + V(\bar{r}) \Rightarrow$

$$\frac{d}{dt}\langle \vec{L} \rangle = \frac{d}{dt}\langle \bar{r} \times \bar{p} \rangle =$$

$$= \frac{i}{\hbar}\langle [H, \bar{r} \times \bar{p}] \rangle =$$

$$= \frac{i}{\hbar}\langle [H, \bar{r}] \times \vec{p} \rangle + \frac{i}{\hbar}\langle \vec{r} \times [H, \vec{p}] \rangle$$

$$= \cancel{\left\langle \frac{\vec{p}}{m} \times \vec{p} \right\rangle}^{0} + \langle \vec{r} \times (-\bar{\nabla} V) \rangle$$

Men $\vec{F} \equiv -\bar{\nabla} V$ och $\bar{r} \times \bar{F} = \vec{N}$

$$\Rightarrow \frac{d\langle \vec{L} \rangle}{dt} = \langle \vec{N} \rangle .$$

b) Om $V(\vec{r}) = V(r)$, dvs sfäriskt symmetrisk potential, så har vi

$$\bar{\nabla} V = \frac{\partial V}{\partial r} \hat{r}$$

där $\hat{r} \equiv \frac{\vec{r}}{|\vec{r}|}$.

Men
$$\vec{N} = -\vec{r} \times \bar{\nabla} V$$
$$= -\vec{r} \times \left(\frac{\partial V}{\partial r} \hat{r} \right)$$
$$= -\frac{\partial V}{\partial r} \bar{r} \times \hat{r} \equiv 0$$

$$\Rightarrow \frac{d\langle \bar{L} \rangle}{dt} = 0$$